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On the resummation of subleading logarithms in the transverse momentum distribution of vector bosons produced at hadron colliders

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Abstract: The perturbation series for electroweak vector boson production at small transverse momentum is dominated by large double logarithms at each order in perturbation theory. An accurate theoretical prediction therefore requires a resummation of these logarithms. This can be performed either directly in transverse momentum space or in impact parameter (Fourier transform) space. While both approaches resum the same *leading* double logarithms, the *subleading* logarithms are, in general, treated differently. We comment on two recent approaches to this problem, emphasising the particular subleading logarithms resummed in each case and the numerical differences in the cross sections which result.

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A complete theoretical description of W and Z boson production at high-energy hadron colliders is necessary for precision Standard Model phenomenology (for example, the measurement of M_W) and for the reliable estimation of backgrounds to new physics processes. An important issue is the calculation of the transverse momentum (q_T) distribution at small q_T, which requires the resummation of large 'Sudakov' double logarithms. These order $\alpha_S^n \ln^{2n-1}(Q^2/q_T^2)$ contributions arise from soft gluon emission and lead to a breakdown of fixed-order perturbation theory as $q_T \to 0$.

Although the resummation of the soft gluon contributions is achieved most naturally in impact parameter (Fourier transform) space [1], there are certain advantages in performing the resummation directly in transverse momentum space [2]. For example, the matching with fixed-order results at large q_T is difficult in the impact parameter approach, since this gives oscillatory behaviour for the large q_T 'tail' of the transformed resummed soft gluon contribution.

The resummation of the large logarithms directly in q_T space has therefore received much recent attention. In particular, two complementary approaches have been proposed in refs. [3] (FNR) and [4] (KS). The aim of both these approaches is to improve the theoretical description of the small q_T cross section by including certain subleading (i.e. order $\alpha_S^n \ln^{2n-r}(Q^2/q_T^2),\ r \geq 2$) logarithms in the resummation. However the resulting predictions are qualitatively very different at very small q_T. The aim of this note is to present and discuss the differences between the FNR and KS calculations.[1] We pinpoint the exact set of subleading logarithms summed in the two cases, and show how their inclusion can lead to significant differences in the predictions.

In order to highlight the differences we will consider both approaches in their most simplified version, i.e. we shall restrict ourselves to the parton-level subprocess cross section and ignore certain other subleading corrections, as decribed below. We also do not consider any non-perturbative (q_T smearing) effects.

The springboard for both approaches is the general expression in impact parameter (b) space for the vector boson transverse momentum distribution in the Drell-Yan process [1] at the quark level:

$$\frac{d\sigma}{dq_T^2} = \frac{\sigma_0}{2} \int_0^\infty bdb\, J_0(q_T b) e^{S(b,Q^2)}\,, \tag{1}$$

where

$$\begin{aligned} S(b,Q^2) &= -\int_{b_0^2/b^2}^{Q^2} \frac{d\bar\mu^2}{\bar\mu^2} \left[\ln\left(\frac{Q^2}{\bar\mu^2}\right) A(\alpha_S(\bar\mu^2)) + B(\alpha_S(\bar\mu^2))\right], \\ A(\alpha_S) &= \sum_{i=1}^\infty \left(\frac{\alpha_S}{2\pi}\right)^i A^{(i)}\,, \qquad B(\alpha_S) = \sum_{i=1}^\infty \left(\frac{\alpha_S}{2\pi}\right)^i B^{(i)}\,, \end{aligned} \tag{2}$$

$$b_0 = 2\exp(-\gamma_E)\,, \qquad \sigma_0 = \frac{4\pi\alpha^2}{9s}\,. \tag{3}$$

[1] We also briefly comment on the related work of [2] (EV).

For the purposes of this simplified analysis we take the coupling α_S be fixed and retain only the leading coefficient $A^{(1)}$, i.e.

$$A(\alpha_S) = \frac{\alpha_S C_F}{\pi}\,, \qquad B(\alpha_S) = 0\,, \tag{4}$$

with $C_F = 4/3$. With these assumptions the $b-$space expression becomes

$$\frac{d\sigma}{dq_T^2} = \frac{\sigma_0}{2}\int_0^\infty b db\, J_0(q_T b)\exp\left(-\frac{\alpha_s C_F}{2\pi}\ln^2\left(\frac{Q^2 b^2}{b_0^2}\right)\right). \tag{5}$$

We recall that the $b-$space formalism takes fully into account the conservation of transverse momentum in multigluon emission, $\delta(\vec{q_T} + \sum_i \vec{k_{Ti}})$, and is therefore expected to provide a better approximation of the q_T distribution at small q_T than the Double Leading Logarithm Approximation (DLLA), in which strong ordering of the gluons' k_{Ti} is assumed. The DLLA leads to the Sudakov form factor expression [5]

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq_T^2}\Big|_{DLLA} = \frac{d}{dq_T^2}\exp\left(-\frac{\alpha_S C_F}{2\pi}\ln^2\left(\frac{Q^2}{q_T^2}\right)\right). \tag{6}$$

The $b-$space expression is (mathematically) well defined for all values of q_T. In particular, in the limit $q_T \to 0$ it gives a finite positive cross section, in contrast to the DLLA result which, because of the vanishing of strong-ordered phase space, yields zero in this limit. However the large q_T behaviour of the $b-$space expression (particularly when higher coefficients $A^{(i)}$, $B^{(i)}$ are taken into account) is not physical – the q_T distribution oscillates between positive and negative values due to the nature of the Bessel function.

After performing partial integration the expression (5) can be written (using the KS notation) as

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta} = \frac{d}{d\eta}\int_0^\infty d\hat{b} J_1(\hat{b})\exp\left(-\frac{\lambda}{2}\ln^2\left(\frac{\hat{b}^2}{\eta b_0^2}\right)\right), \tag{7}$$

with $\hat{b} = b q_T$, $\lambda = \alpha_S C_F/\pi$, $\eta = q_T^2/Q^2$.

The next step in the KS approach is to expand terms under the integral in (7), which gives

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta} = \frac{1}{\eta}\sum_{N=1}^\infty \lambda^N \frac{(-1/2)^{N-1}}{(N-1)!}\sum_{m=0}^{2N-1} 2^m \tau_m \binom{2N-1}{m} L^{2N-1-m}\,. \tag{8}$$

Here $L = \ln(1/\eta) = \ln(Q^2/q_T^2)$ and the numbers τ_m are defined by[2]

$$\tau_m \equiv \int_0^\infty dy J_1(y)\ln^m\left(\frac{y}{b_0}\right) \tag{9}$$

[2]The τ_m are called $\bar{b}_m(\infty)$ in [4].

and can be calculated explicitly using the generating function

$$\sum_{m=0}^{\infty} \frac{1}{m!} t^m \tau_m = e^{t\gamma_E} \frac{\Gamma(1+t/2)}{\Gamma(1-t/2)} = \exp\left[-2\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1} \left(\frac{t}{2}\right)^{2k+1}\right], \tag{10}$$

so that e.g. $\tau_0 = 1$, $\tau_1 = \tau_2 = 0$, $\tau_3 = -\frac{1}{2}\zeta(3)$ etc. For large m, the coefficients τ_m behave as $\tau_m \propto (-1)^m m! 2^{-m}$, and the first twenty τ_m are tabulated in [4].

By extracting the Sudakov form factor $\exp\left(-\frac{\lambda}{2}L^2\right)$ the expression (8) transforms into

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta} = \frac{\lambda}{\eta} e^{-\frac{\lambda}{2}L^2} \sum_{N=1}^{\infty} \frac{(-2\lambda)^{(N-1)}}{(N-1)!} \sum_{m=0}^{N-1} \binom{N-1}{m} L^{N-1-m} \Big[2\tau_{N+m} + L\tau_{N+m-1}\Big]. \tag{11}$$

Naturally, for numerical calculations based on the expression (11) it is necessary to introduce a cut-off value $N_{\max}$. For example, figure 1 shows the contributions to the double summation in (11) which are summed when $N_{\max} = 4$. Some illustrative numerical results based on the KS approach will be presented below.

Another approach, suggested in [2] (EV), succeeds in developing an *analytic* approximation when a slightly modified set of assumptions is considered. That is to say, all the τ_m coefficients except τ_0 are set to be zero and $B^{(2)}$ acquires an additional contribution $-4\tau_3(A^{(1)})^2 = 2\zeta(3)(A^{(1)})^2$. Including the expanded Sudakov factor, see figure 2, this corresponds to fully summing the first three leading series of logarithms, i.e. terms of the form $\alpha_S^N L^{2N-1-m}$, $m = 0, 1, 2$. In the EV approach, the redefinition of $B^{(2)}$ correctly takes account of the *first* term of the fourth series, i.e. the $\alpha_S^2 L^0$ term. On the other hand, it distorts other terms of this series and terms from more subleading series wherever the $B^{(2)}$ coefficient appears.

In the FNR approach [3], one expands the exponent in (7)

$$\exp\left(-\frac{\lambda}{2}\ln^2\left(\frac{\hat{b}^2}{\eta b_0^2}\right)\right) = \exp\left(-\frac{\lambda}{2}\left(L^2 + 2L\,L_b + L_b^2\right)\right) \tag{12}$$

where $L_b = \ln(\hat{b}^2/b_0^2)$, and retains only the first two terms ('NLL approximation'):

$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{[3]} = \frac{d}{d\eta}\int_0^{\infty} d\hat{b} J_1(\hat{b}) \exp\left(-\frac{\lambda}{2}L^2 - \lambda L\,L_b\right). \tag{13}$$

Note that keeping only the leading $\sim L^2$ term in the exponent corresponds to the DLLA. A great advantage of the 'NLL approximation' is that the $\hat{b}$ integral can be performed analytically:[3]

$$\left.\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\right|_{[3]} = \frac{d}{d\eta}\left[\exp\left(-\frac{\lambda}{2}L^2\right)\left(\frac{2}{b_0}\right)^{-2\lambda L} \frac{\Gamma(1-\lambda L)}{\Gamma(1+\lambda L)}\right]. \tag{14}$$

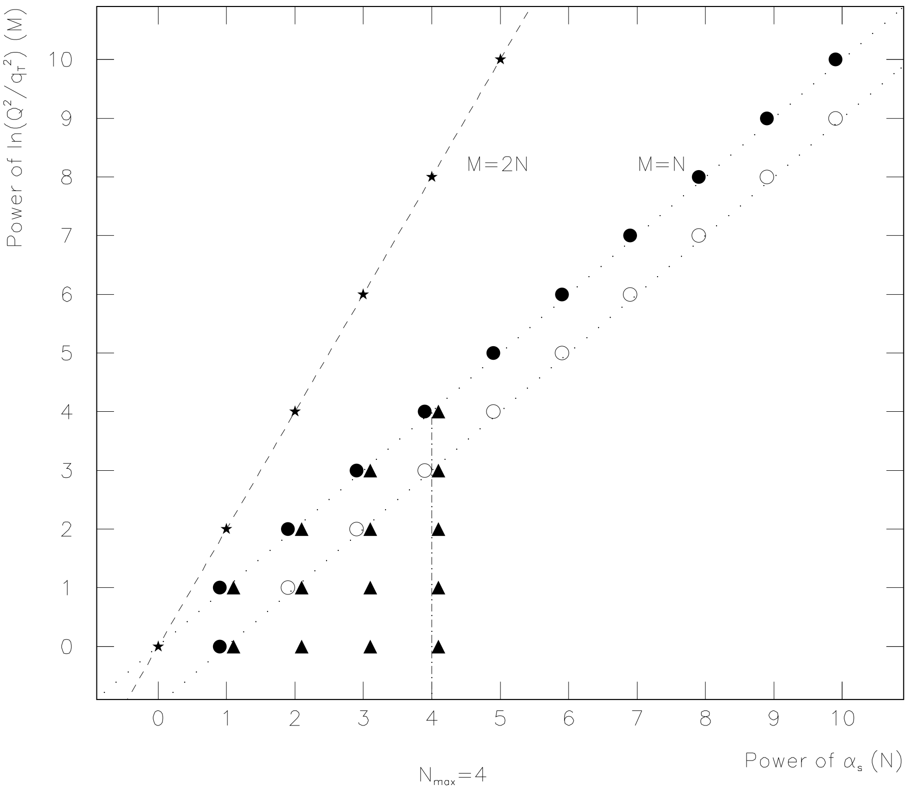

Figure 1: Schematic representation of contributions to (15) and to (11). Circles correspond to the former expression, triangles to the latter one. An empty marker of a certain shape means that there exist other contributions in the perturbation series with the same power of α_S and $\ln(Q^2/q_T^2)$ which are not included in an expression coded with that shape. The points along the line labelled '$M = 2N$' represent terms coming from the Sudakov factor.

There is in fact a direct link between the KS and FNR approaches. If instead of performing the integration in (13) one expands the b-dependent terms in the exponent and then performs the integration, the result is

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\Big|_{[3]} = \frac{\lambda}{\eta} e^{-\frac{\lambda}{2}L^2} \sum_{N=1}^{\infty} \frac{(-2\lambda)^{(N-1)}}{(N-1)!} L^{N-1} \left[2\tau_N + L\tau_{N-1}\right]. \tag{15}$$

Clearly this is just the expression (11) taken at $m = 0$. Indeed the same result can be derived from the resummed expression (14) by recalling the definition of the generating function (10) and using the relation

$$\ln\Gamma(1+x) = -\ln(1+x) + x(1-\gamma_E) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{x^n}{n}, \qquad |x| < 2. \tag{16}$$

The contributions being resummed in both approaches are illustrated schematically in figure 1 and figure 2. The aim of figure 1 is to show which terms of the form

[3]Note that throughout this note the value of the lower limit of integration in (2) is b_0^2/b^2. This is different from [3] where $1/b^2$ is chosen. Therefore the expression (14) differs from the original expression in [3] by a constant.

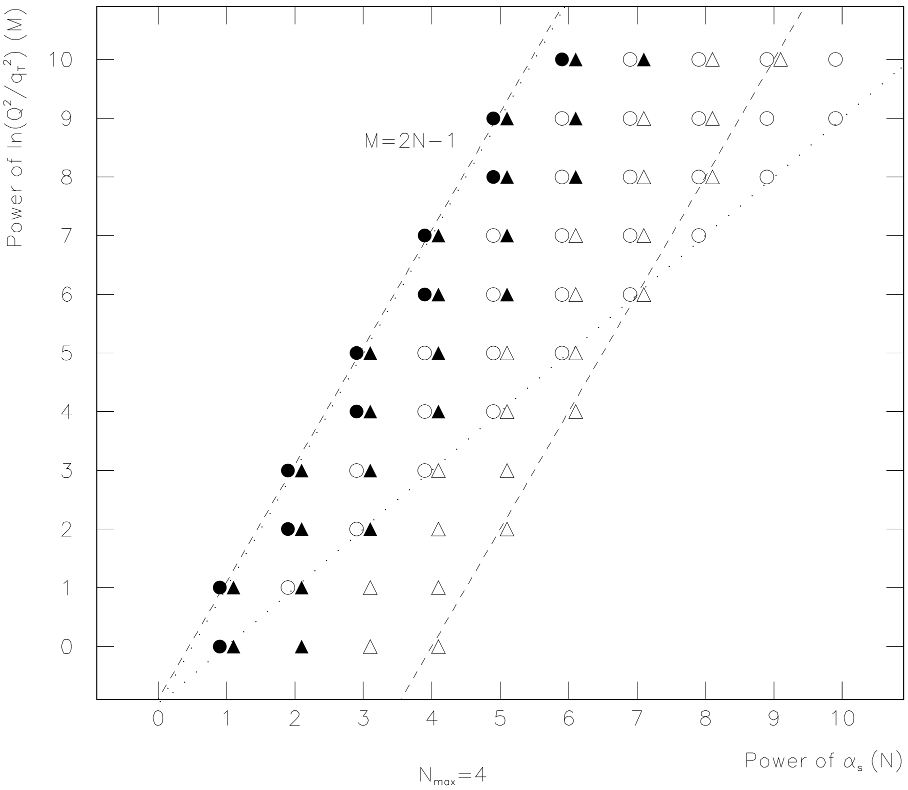

Figure 2: Schematic representation of contributions to (15) with the Sudakov factor expanded and (8). Circles correspond to the former expression, triangles to the latter one. An empty marker of a certain shape means that there exist other contributions in perturbation theory of the same power of α_S and $\ln(Q^2/q_T^2)$ which are not included in an approach coded with that shape.

$\alpha_S^N L^M$ from the residual sums in (11) and (15) are taken into account. Here the FNR approach corresponds to having two infinite lines of points (terms) while the KS approach results in a finite triangle of terms with size determined by $N_{\max}$. Terms emerging in the *full* perturbative expansion, i.e. after expanding and multiplying in the Sudakov factor, in both approaches are illustrated in figure 2. Note that summing over all logarithmic terms with a given power of α_S must result in the perturbative expansion coefficient of the same order, up to logarithmic accuracy. Of course a formula with an expanded Sudakov factor is valid only when $\alpha_S L^2 \lesssim 1$. The only reason for expanding the Sudakov factor here is to determine which terms in the overall perturbation series are actually being resummed in (11) and (14). It is these latter expressions, which can be regarded as the 'master equation' of the two approaches, that we use to obtain numerical results, and both approaches remain well-behaved provided $\alpha_S L \lesssim 1$.

We next present some simple numerical comparisons using both approaches.[4] First we investigate the dependence of the KS result on the point of truncation

[4]The value of the strong coupling constant has been fixed here at $\alpha_S = 0.2$.

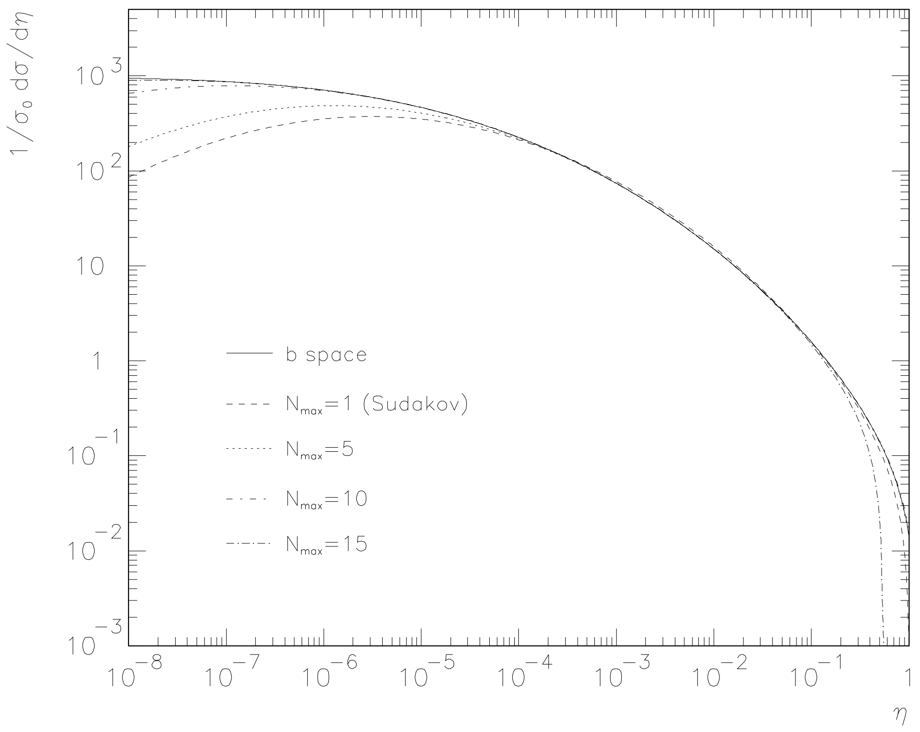

Figure 3: The $b-$space result compared to the expression (11), calculated for various values of $N_{\rm max}$. Here $N_{\rm max} = 1$ corresponds to the DLLA approximation.

$N_{\rm max}$. (Note that $N_{\rm max} = 1$ is just the DLLA approximation.) It is clear from figure 3 that for small values of η (i.e. $q_T/Q \ll 1$) the approximating of the $b-$space result improves with increasing $N_{\rm max}$. The 'Sudakov dip' at very small η is more and more filled in as $N_{\rm max}$ increases, by contributions which are formally subleading in terms of powers of α_S and L.

On the other hand, the range of applicability of the FNR resummed formula (14) is seriously restricted. As pointed out in [3], the expression (14) suffers from singularities at $\lambda L = 1, 2, \ldots$ (In fact these singularities are poles of order two.) Therefore the first pole encountered as η decreases is at $\eta^{\rm crit} = \exp(-1/\lambda)$, i.e. $q_T^{\rm crit} = Q\exp(-\pi/2\alpha_S C_F)$.[5] Figure 4 shows the resummed FNR result (14) as a function of η. The pole at $\eta^{\rm crit}$ is evident (the distribution $\to -\infty$ as the singularity is approached from above). The resummed result is also compared to the 'truncated' expression (15) for various values of the cut-off parameter $N_{\rm max}$. This shows the effect of successively adding more and more of the subleading '$m = 0$' terms along the two infinite lines of figure 1, starting from the Sudakov expression for $N_{\rm max} = 1$. Convergence to the (singular) resummed FNR result (14) for large $N_{\rm max}$ is clearly evident.

A natural extension of the approach of [3] would be a resummed analytic expression also including $m = 1$ terms in the classification of (15). In fact one can

[5]This may appear to be an irrelevantly small value but, as shown in [3], when the running coupling constant is used the pole moves significantly towards higher values of q_T.

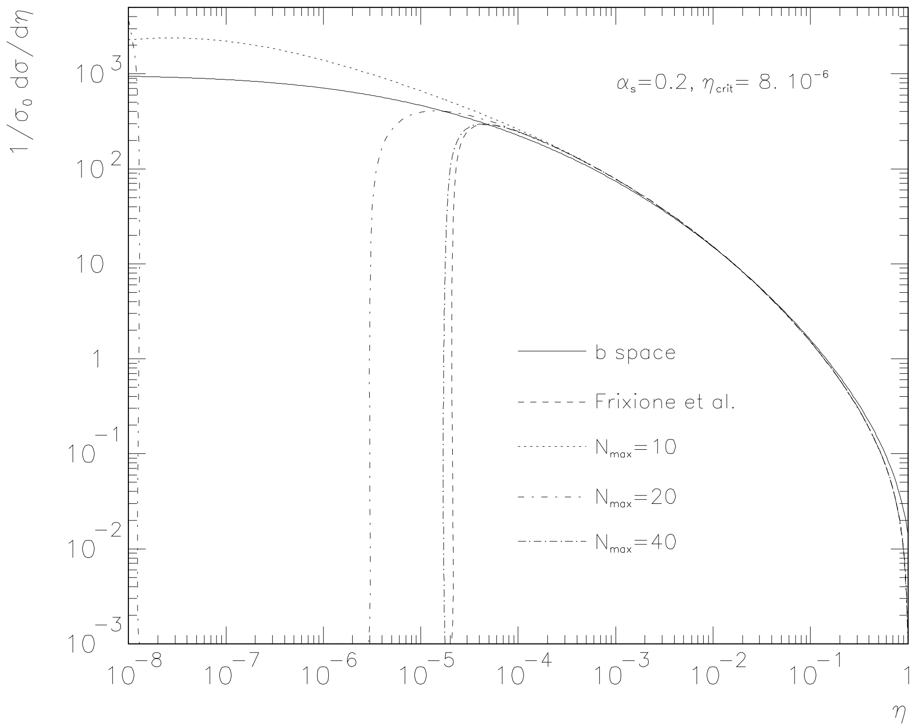

Figure 4: The $b-$space result compared to the expression (14)and (15), calculated for various values of $N_{\max}$. With the choice $\alpha_s = 0.2$, (14) is only applicable for $\eta \gtrsim 8. \times 10^{-6}$.

systematically include the subleading 'NNLL' terms of (12) using the identity

$$\exp\left(-\frac{\lambda}{2}\left(2L\,L_b\,+\,L_b^2\right)\right) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(-\frac{\lambda}{2}\right)^j \frac{d^{2j}}{d(\lambda L)^{2j}}\,\exp\left(-\lambda L\,L_b\right) \tag{17}$$

which generates more subleading terms as derivatives of the FNR analytic 'NLL' result. In particular, including the $m = 0, 1$ contributions yields

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\eta}\Big|_{m=0,1} = \frac{d}{d\eta}\left\{\exp\left(-\frac{\lambda}{2}L^2\right)\left(1 - \frac{\eta^2}{2\lambda}\frac{d^2}{d^2\eta} - \frac{\eta}{2\lambda}\frac{d}{d\eta}\right)\left[\left(\frac{2}{b_0}\right)^{-2\lambda L}\frac{\Gamma(1-\lambda L)}{\Gamma(1+\lambda L)}\right]\right\}. \tag{18}$$

This, however, does not cure the singularity problem but makes it even worse. It turns out that if the upper limit of the sum over m increases by 1, the order of the poles increases by 2, e.g. the formula (18) has poles of order four at $\eta^{\rm crit}$. Moreover, as this upper limit increases, the region where the approximation of the $b-$space result becomes better, contracts. This is illustrated in figure 5, where we show the ratio of the numerically calculated expression (14), (18) and the 'full' $b-$space result.

The authors of [3] argue that the subleading terms in the original expression (7) possess a divergent behaviour. These terms have been shown to have factorially growing coefficients and originate from the small-$\hat{b}$ region of integration. Their presence manifests itself in the KS approach which takes some of those subleading terms

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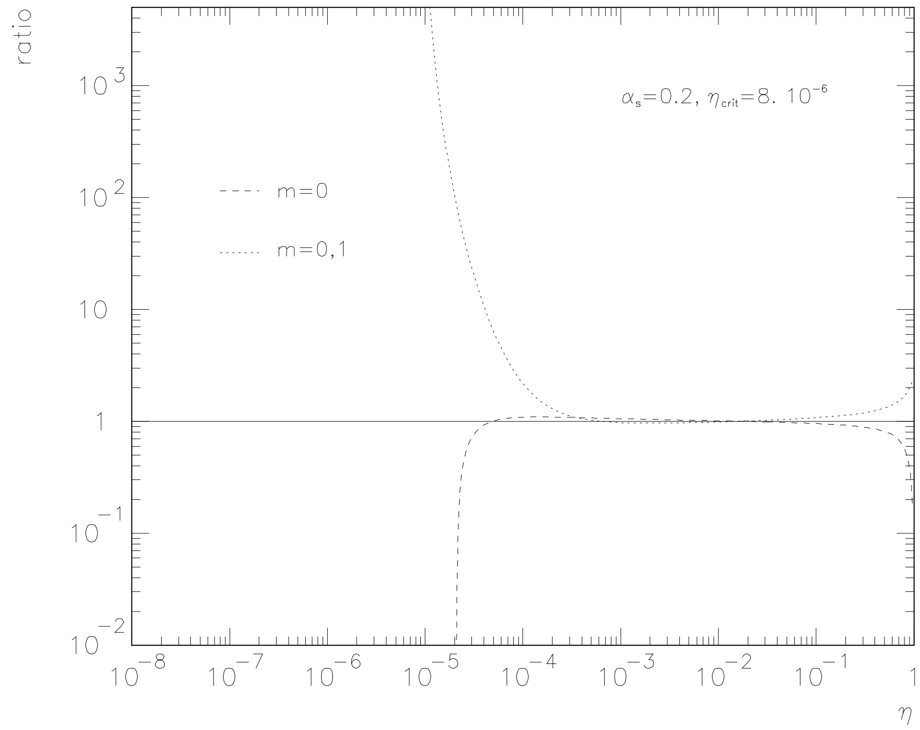

Figure 5: The ratio of the numerically calculated (14) (m=0 curve) and (18) (m=0,1 curve) to the $b-$space result.

into account. Indeed for large $N_{\rm max}$ (i.e. more subleading terms), the presence of factorially growing subleading coefficients can be seen at large η, e.g. in figure 3, where the cross section can become negative. But this behaviour is not entirely unexpected. Let us recall that the $b-$space formalism was invented to provide a good description in the *small* η regime where $\alpha_S \ln(1/\eta)$ becomes large. Thus the presence of factorially growing terms which manifest themselves for large η can be understood as an artefact of the $b-$space method. In fact, in this formalism, the recovery of a credible theoretical result in the large η domain relies on careful matching with fixed-order perturbation theory.

In summary, we have shown that the KS and NFR approaches start from the same expression for the cross section in b space, but organise the perturbative expansion in q_T space in different ways such that different subleading q_T logarithms are included. In the KS approach, 'towers' of subleading logarithms fill in the Sudakov (DLLA) dip at small q_T. In the FNR approach, a particular 'subset' of subleading logarithms is resummed to all orders, but the resulting expression has a singularity at $q_T = q_T^{\rm crit}$, below which the cross section is not defined. The FNR result can be obtained in the KS approach by including only the appropriate subleading terms.

The basic question remains as to whether there is a definite value of q_T below which the perturbative expression for the cross section cannot be calculated. The original argument for the b-space approach was that it allowed a non-zero cross section at $q_T = 0$ to be generated by the emission of soft gluons whose transverse momentum vectors cancelled, a phase space region clearly outside the strongly-ordered DLLA domain. The KS approach is designed to take these (formally subleading) contributions into account in a systematic way. The 'price' one pays is a series with factorially growing coefficients that drive the behaviour at large q_T, but this is in any case outside the region of applicability of the whole approach. In contrast, the validity of the FNR approach seems to depend on the extent to which the 'LL' and 'NLL' terms as defined in eq. (12) do actually give the dominant contribution to the small q_T cross section. Since we have shown that attempts to systematically include the 'NNLL' contributions in this approach lead to even more singular behaviour than the one observed in the 'NLL' case, this may cast doubt on the validity of the 'NLL' approximation. In any case, one incontrovertible conclusion is that this is an interesting and important issue that deserves further study.

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